Problem 1

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Problem 1. (5 + 10 + 10 = 25 points) *This is Exercise 5.4.3, before using it as a homework I planned it as an exam exercise as follows.*

For this problem you cannot use the results in Chapter 8.2 and 8.3. We are going to show that line integrals are well defined and generalize the fundamental theorem of calculus.

Let M be a smooth manifold, $\gamma : I = [a,b] \subset \mathbb{R} \to M$ a smooth curve and $\omega \in \mathfrak{X}^*(M)$ a 1-form. Show the following properties.

(a) Show that for line integrals

$$\int_\gamma \omega = \int_a^b \omega_{\gamma(t)}(\gamma'(t))\,dt. \tag{1}$$

(b) Let $J \subset \mathbb{R}$ be an open interval and $F : J \to I$ an orientation-preserving diffeomorphism. Show that

$$\int_{F^*\gamma} \omega = \int_\gamma \omega. \tag{2}$$

Hint: use the chain rule to get $(F^*\gamma)'(t) = \gamma'(F(t))F'(t)$ and then apply (1).

(c) Let $f \in C^\infty(M)$. Prove the fundamental theorem of calculus:

$$\int_\gamma df = f(\gamma(b)) - f(\gamma(a)). \tag{3}$$

Hint: reduce the formula to the usual fundamental theorem of calculus on $\mathbb{R}$.

(a) $\int_\gamma \omega := \int_I \gamma^*\omega := \int_a^b \left(\gamma^*\omega \mid \frac{\partial}{\partial t}\right)(t)\,dt$ $\boxed{1}$ Recall the definition of line integral.

By definition, for a curve $\gamma : [a,b] \longrightarrow M$, we have $\gamma'(t) \in T_{\gamma(t)}M$, defined by $\gamma'(t) = d\gamma_t\left(\frac{\partial}{\partial t}\Big|_t\right)$ $\boxed{2}$

Moreover, by definition of the pull-back:

$$\left(\gamma^*\omega \mid \frac{\partial}{\partial t}\right)(t)$$
$$= \omega_{\gamma(t)}\left(d\gamma_t\left(\frac{\partial}{\partial t}\Big|_t\right)\right)$$
$$= \omega_{\gamma(t)}\left(\gamma'(t)\right) \quad \boxed{2}$$

hence the claim follows. $\square$

(b) $F : J \to I = [a,b]$. Let's say $J = [c,d]$.

$$\int_{F^*\gamma} \omega = \int_c^d \omega_{F^*\gamma(t)}\left((F^*\gamma)'(t)\right)dt \quad \boxed{2}$$

By the chain rule: $(F^*\gamma)'(t) = (\gamma \circ F)'(t)$
$= \gamma'(F(t))\ F'(t)$ $\boxed{3}$

Hence, $\int \omega = \int^d \omega_{\ldots} \left(F'(t)\gamma'(F(t))\right)dt$

Hence, $\int_{F^*\gamma} \omega = \int_c^d \omega_{\gamma(F(t))} (F'(t) \gamma'(F(t)) dt$

$= \int_c^d \omega_{\gamma(F(t))} (\gamma'(F(t))) F'(t) dt$ $\boxed{3}$

Let $u = F(t)$, hence $du = F'(t) dt$

$\overset{\circledast}{=} \int_a^b \omega_{\gamma(u)} (\gamma'(u)) du = \int_\gamma \omega$ $\square$

$\circledast$ here, we use that F preserves orientation. otherwise, the integral would pick up a $\ominus$ sign. $\boxed{2}$

(c) $\int_\gamma df = \int_a^b df_{\gamma(t)} (\gamma'(t)) dt$

definition γ' $\boxed{2}$

$= \int_a^b df_{\gamma(t)} \, d\gamma_t \left(\frac{\partial}{\partial t}\Big|_t \right) dt$

chain rule $\boxed{3}$

$= \int_a^b d(f \circ \gamma)_t \left(\frac{\partial}{\partial t}\Big|_t \right) dt$

definition of differential $\boxed{3}$

$= \int_a^b \frac{\partial}{\partial t}\Big|_t (f \circ \gamma) \, dt$

$= (f\circ\gamma)(b) - (f\circ\gamma)(a)$ FTC on $\mathbb{R}$ $\boxed{2}$ $\square$

Problem 2. (7 + 10 + 10 + 5 = 32 points)
The goal of this exercise is to show that the three sphere is parallelizable without using the theory of Lie groups.
Consider $\mathbb{S}^3 = \{(x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + w^2 = 1\} \subset \mathbb{R}^4$.

(a) Show that $T_p\mathbb{S}^3$ as a subspace of $T_p\mathbb{R}^4$ is identified with $p^\perp := \{q \in \mathbb{R}^4 \mid p \cdot q = 0\}$.

(b) Show that
$$X = w\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} - x\frac{\partial}{\partial w}$$
is tangent to $\mathbb{S}^3$.

(c) Find another vector field Y, given by a similar formula, that is also tangent to $\mathbb{S}^3$ and such that X, Y and $Z := [X, Y]$ span the tangent space $T_p\mathbb{S}^3$ for all $p \in \mathbb{S}^3$.

(d) Show that $T\mathbb{S}^3$ is trivializable.

(a) Take $F: \mathbb{R}^4 \to \mathbb{R}$, $F(x,y,z,w) = x^2+y^2+z^2+w^2-1$.
$\Rightarrow \mathbb{S}^3 = F^{-1}(0)$.

(Note that 0 is a regular value of F, and $F^{-1}(0)$ is non-empty. So, $\mathbb{S}^3$ is a 3-dimensional submanifold of $\mathbb{R}^4$)

By Prop 2.8.23, $T_p\mathbb{S}^3 \subseteq T_p\mathbb{R}^4$ should be identified with the kernel of dF_p. (After applying τ_p) $\boxed{3}$
Realizing that 2.8.23 should be used.

$$dF_p = 2x\,dx_p + 2y\,dy_p + 2z\,dz_p + 2w\,dw_p \quad (p = (x,y,z,w))$$

Applied to a tangent vector $q = a\frac{\partial}{\partial x}\Big|_p + b\frac{\partial}{\partial y}\Big|_p + c\frac{\partial}{\partial z}\Big|_p + d\frac{\partial}{\partial w}\Big|_p$ gives $2ax + 2by + 2cz + 2dw = 0$

$$\Leftrightarrow \underbrace{\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}}_{=\vec{q}} \cdot \underbrace{\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}}_{=\vec{p}} = 0$$

after some canonical identification.

Thus, $T_p\mathbb{S}^3 = \left\{ \vec{q} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; \vec{q}\cdot\vec{p} = 0 \right\} =: \vec{p}^{\perp}$. $\boxed{4}$ $\square$
Computation $\frac{}{5}$

(b) X is identified with the vector $\begin{pmatrix} w \\ z \\ -y \\ -x \end{pmatrix}$ in $\mathbb{R}^4$. $\boxed{5}$

We have $\begin{pmatrix} w \\ z \\ -y \\ -x \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = xw + yz - yz - xw = 0$,
so X is tangent to $\mathbb{S}^3$. $\boxed{5}$

(c) Swapping two components and adding a $\ominus$ sign always will give something tangent to $\mathbb{S}^3$.

always will give something tangent to $\mathbb{S}^3$.

Probably the best way to solve this exercise is just by trial and error.

Choosing a proper Y $\boxed{2}$

We take $Y = \begin{pmatrix} z \\ -w \\ -x \\ y \end{pmatrix}$. This is tangent to $\mathbb{S}^3$ by the same computation as in part (b). $\boxed{1}$

Compute $[X,Y]$ by definition or using formula $[X,Y] = \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}$.

Fill this in with $X = (w, z, -y, -x)^T$, $Y = (z, -w, -x, y)^T$

$$[X,Y] = \left(w \frac{\partial}{\partial x} + z \frac{\partial}{\partial y} + (-y) \frac{\partial}{\partial z} + (-x) \frac{\partial}{\partial w} \right) \left(z \frac{\partial}{\partial x} + (-w) \frac{\partial}{\partial y} + (-x) \frac{\partial}{\partial z} + y \frac{\partial}{\partial w} \right)$$
$$- \left(z \frac{\partial}{\partial x} + (-w) \frac{\partial}{\partial y} + (-x) \frac{\partial}{\partial z} + y \frac{\partial}{\partial w} \right) \left(w \frac{\partial}{\partial x} + z \frac{\partial}{\partial y} + (-y) \frac{\partial}{\partial z} + (-x) \frac{\partial}{\partial w} \right)$$
$$= w \frac{\partial}{\partial x}(-x) \frac{\partial}{\partial z} + z \frac{\partial y}{\partial y} \frac{\partial}{\partial w} + (-y) \frac{\partial z}{\partial z} \frac{\partial}{\partial x} + (-x) \frac{\partial(-w)}{\partial w} \frac{\partial}{\partial y}$$
$$- z \frac{\partial(-x)}{\partial x} \frac{\partial}{\partial w} + w \frac{\partial(-y)}{\partial y} \frac{\partial}{\partial z} + x \frac{\partial z}{\partial z} \frac{\partial}{\partial y} - y \frac{\partial w}{\partial w} \frac{\partial}{\partial x}$$

(All cross terms cancel)

Computation of Z $\boxed{4}$

$$= (-2y,\ 2x,\ -2w,\ 2z)$$

Again, we note that $Z = [X,Y]$ is tangent to $\mathbb{S}^3$, $\boxed{1}$ and

$$\begin{pmatrix} w \\ z \\ -y \\ -x \end{pmatrix} \quad \begin{pmatrix} z \\ -w \\ -x \\ y \end{pmatrix} \quad \begin{pmatrix} -y \\ x \\ -w \\ z \end{pmatrix}$$

are linearly independent at each point, since they are mutually orthogonal, and nonzero on $\mathbb{S}^3$. Thus, they form a basis for $T_p\mathbb{S}^3$. (A global frame). $\boxed{2}$

(d) X, Y, Z form a global frame, so by proposition 2.7.13, $T\mathbb{S}^3$ is trivializable. (They span the tangent space at each point, and vanish nowhere). $\boxed{5}$

Problem 3. **(8 + 10 + 5 + 10 = 33 points)** Let V a vector space of dimension k. A symplectic form on V is an element $\omega \in \Lambda^2(V)$ which is non-degenerate in the sense that $\iota_v(\omega) = 0$ if and only if $v = 0$.
A *symplectic manifold* is a smooth manifold M equipped with a closed differential 2-form ω such that ω_q is a symplectic form on T_qM for every $q \in M$.

(a) Prove that if a symplectic form exists, then $k = 2n$ for some $n \in \mathbb{N}$, i.e., it must be an even number.

(b) Let M be a smooth manifold. Define a 1-form $\eta \in \Omega^1(T^*M)$ on the cotangent bundle of M as

$$\lambda_{(q,p)} = d\pi^*_{(q,p)} p, \quad q \in M,\ p \in T_q^*M, \tag{4}$$

where $\pi : T^*M \to M$ is the projection to the base. If (x^i, ξ_i) denote the local coordinates induced by a chart of M on T^*M, show that $\lambda_{(x,\xi)} = \xi_i dx^i$.

(c) Show that $\omega := d\lambda$ is a symplectic form on T^*M, that is, every cotangent bundle is a symplectic manifold.

(d) Use $\omega = d\lambda$ to show that T^*M is orientable.

(a) $\omega \in \Lambda^2(V)$ is anti-symmetric and non-degenerate.

Pick a basis $\{e_1, \dots, e_n\}$. $\Rightarrow$ $\omega(e_i, e_j) =: \omega_{ij}$ represents ω as a matrix. $\left(\omega(v_1^i e_i, v_2^j e_j) = v_1^i v_2^j \omega_{ij}\right.$ $\boxed{2}$
$= \vec{v}_1^T \omega \vec{v}_2$

Note that ω is an anti-symmetric matrix $(\omega_{ij} = -\omega_{ji})$ $\boxed{2}$,
and $\det(\omega) \neq 0$. $\boxed{2}$ $\left(\det(\omega) = 0 \Rightarrow \omega\vec{v} = 0\right.$ for some nontrivial $\vec{v} \in V$ $\Rightarrow \iota_{\vec{v}}\omega = 0$ ↯ $)$

But $\det(\omega) = \det(\omega^T) = \det(-\omega) = (-1)^{\dim V} \det(\omega)$
So, if $\dim(V)$ is odd, then $\det(\omega)$ must be zero. ↯. $\boxed{2}$
Thus, $\dim(V)$ must be even. □

(b) $\lambda_{(q,p)} = d\pi^*_{(q,p)} p$. $q \in M$, $p \in T_q^*M$

$\pi : T^*M \to M$ projection to the base

p : 1-form on M

Let (x^i, ξ_i) be the local coordinates on T^*M.

$d\pi^*(dx^i) = d(x^i \circ \pi) = dx^i$ (since $\pi : (x, \xi) \longmapsto x$) Apply property of π $\boxed{5}$

$$\Rightarrow \quad d\pi^*(\xi_i dx^i) = \underline{\underline{\xi_i dx^i}}.$$ $\boxed{5}$ Apply property of pullback.

(c) $\omega = d\lambda$ symplectic form on T^*M.

- ω is closed, since $d\omega = d(d\lambda) = 0$. $\boxed{1}$
- In local coordinates: $\underline{\lambda = d\xi_i \wedge dx^i}$ which is non-degenerate. $\boxed{2}$

on the basis $\left\{ \frac{\partial}{\partial \xi_j}, \frac{\partial}{\partial x^j} \right\}$:

$$\iota_{\frac{\partial}{\partial \xi_j}}(\lambda)\left(\frac{\partial}{\partial x^k}\right) = \begin{vmatrix} d\xi_i\left(\frac{\partial}{\partial \xi_j}\right) & d\xi_i\left(\frac{\partial}{\partial x^k}\right) \\ dx^i\left(\frac{\partial}{\partial \xi_j}\right) & dx^i\left(\frac{\partial}{\partial x^k}\right) \end{vmatrix}$$

$$= \begin{vmatrix} \delta^i_j & 0 \\ 0 & \delta^i_k \end{vmatrix}$$

which is nonzero if you choose $j = i$ and $k = i$. Hence

$$\iota_{\frac{\partial}{\partial \xi_j}} \lambda \neq 0 \quad \forall j.$$ $\boxed{2}$

Thus, T^*M is a symplectic manifold. $\square$

(d). T^*M is orientable $\iff$ $\exists$ nowhere vanishing $2n$-form.

Take $$\eta = \underbrace{\lambda \wedge \lambda \wedge \dots \wedge \lambda}_{n \text{ times}}$$ $\boxed{5}$ right intuition

$$\eta = (d\xi_{i_1} \wedge dx^{i_1}) \wedge \dots \wedge (d\xi_{i_n} \wedge dx^{i_n})$$

$$= \sum_{\substack{i_1 \dots i_n \\ \text{all different}}} (d\xi_{i_1} \wedge dx^{i_1}) \wedge \dots \wedge (d\xi_{i_n} \wedge dx^{i_n})$$

$$= \sum_{\sigma \in S_n} (d\xi_{\sigma(1)} \wedge dx^{\sigma(1)}) \wedge \dots \wedge (d\xi_{\sigma(n)} \wedge dx^{\sigma(n)})$$

$$= n!\,(d\xi_1 \wedge dx^1 \wedge \dots \wedge d\xi_n \wedge dx^n.)$$ $\boxed{5}$ computation.

This is clearly a nowhere zero top-form. $\square$